A special case of the data arrangement problem on binary trees

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Short overview

- problem definition

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- upper bound

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- problem definition
- upper bound (solution algorithm)

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- problem definition
- upper bound (solution algorithm)
- lower bound

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- lower bound:
    - problem transformation

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- recapitulation, future research and open questions

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$$\sum_{(i,j)\in E(G)} d\big(\phi(i), \phi(j)\big), \tag{1}$$

  where $d(x, y)$ denotes the length of the shortest path between x and y in T.

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- In our case T is a d-regular tree and B is the set of its leaves.
- We will call this problem **data arrangement problem on regular trees (DAPT)** and denote the objective value $OV(G, d, \phi)$.

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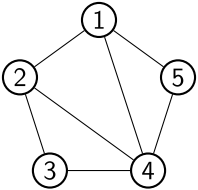

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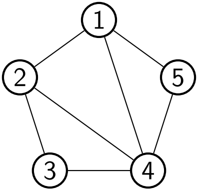

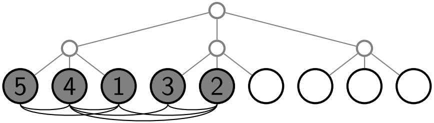

$$OV(G,3,\phi) = 20$$

General properties and our special case

- DAPT is $\mathcal{NP}$-hard for every fixed $d \geq 2$ [Luczak, Noble 2002[4]].

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- We deal with the special case where G and T are both binary regular trees.

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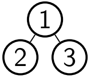
Solution algorithm



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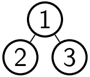
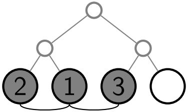
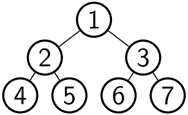




$$OV(G, 2, \phi^*) = 6$$

Solution algorithm



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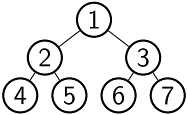
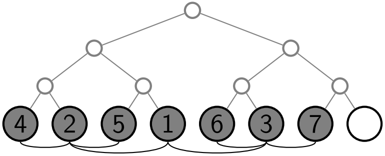
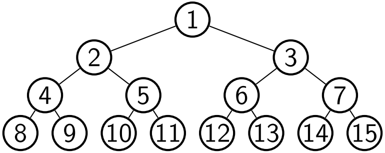




$$OV(G, 2, \phi^*) = 22$$

Solution algorithm



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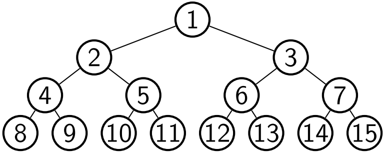
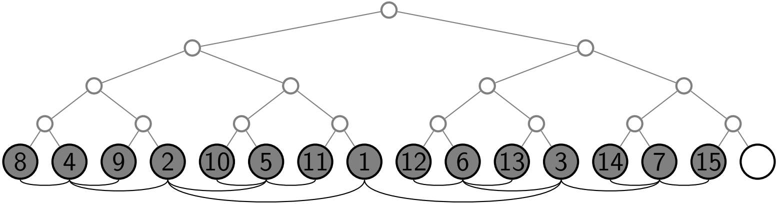
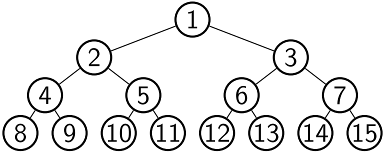
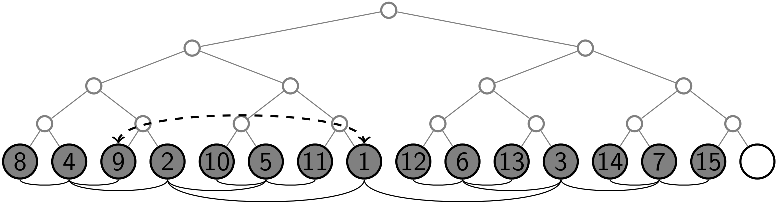




$$OV(G, 2, \phi^*) = 58$$

Solution algorithm





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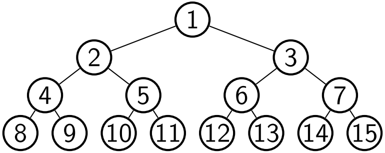
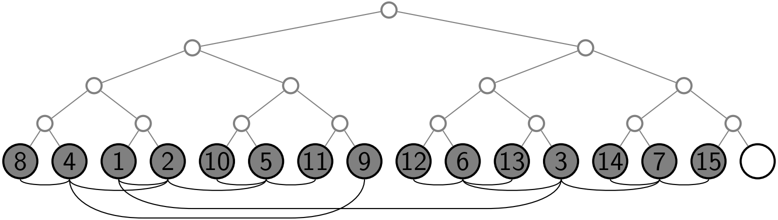




$$OV(G, 2, \phi^*) = 56$$

Solution algorithm

```
Require: binary regular tree $G = (V, E)$ of height $h_G$ labelled according
    to the canonical order
Ensure: arrangement $\phi^*$
$b := 2^{h_G+1}$;
if $h_G = 0$ then
    $\phi^*(v_1) := b_1$;
else $\{h_G > 0\}$
    solve the problem for the basic subtrees $\widehat{G_1}$ and $\widehat{G_2}$, place the
    obtained arrangements on the leaves $b_1, b_2, \ldots, b_{\frac{1}{2}b}$ and $b_{\frac{1}{2}b+1}$,
    $b_{\frac{1}{2}b+2}, \ldots, b_b$ and, finally, place the root on the leaf $b_{\frac{1}{2}b}$;
    if $h_G$ is odd and $h_G \geq 3$ then
        make pair-exchange of the vertices arranged on the leaves $b_{\frac{1}{4}b-1}$
        and $b_{\frac{1}{2}b}$;
    end if
end if
return $\phi^*$;
```

Solution algorithm

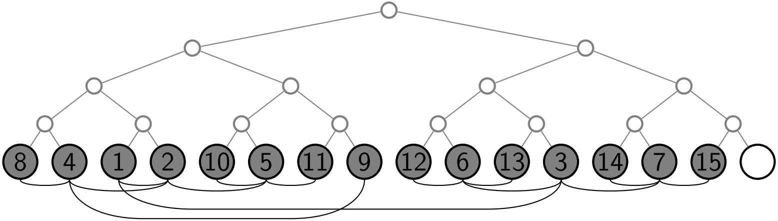
Theorem

Given the binary regular trees $G = (V, E)$ and T with heights h_G and $h = h_G + 1$, let G be the guest graph and T the host graph and let ϕ^* be the arrangement obtained from the described algorithm. Then

$$OV(G, 2, \phi^*) = \begin{cases} 0 & \text{for } h_G = 0 \\ \frac{29}{3} \cdot 2^{h_G} - 4h_G - 9 + \frac{1}{3}(-1)^{h_G} & \text{for } h_G \geq 1 \end{cases} \tag{2}$$

holds.

Lower bound – problem transformation



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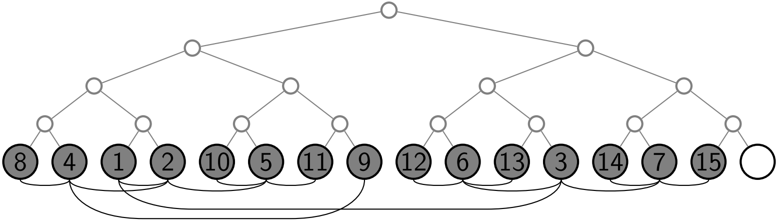

$$OV(G, 2, \phi^*) = 56$$

- $OV(G, 2, \phi) = 2(1 \cdot 4 + 3 \cdot 3 + 5 \cdot 2 + 5 \cdot 1) = 56$

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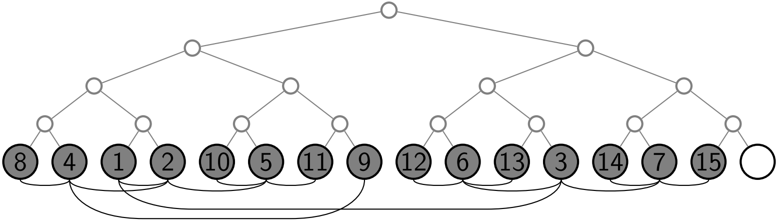

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- $OV(G, 2, \phi) = 2\big(a_h(\phi) \cdot h + a_{h-1}(\phi) \cdot (h-1) + \ldots + a_1(\phi) \cdot 1\big)$

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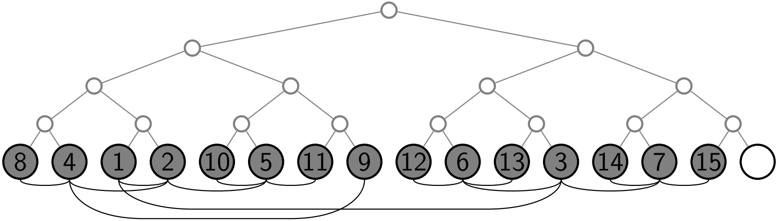

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- $OV(G, 2, \phi) = 2 \sum_{i=1}^{h} a_i(\phi) \cdot i$

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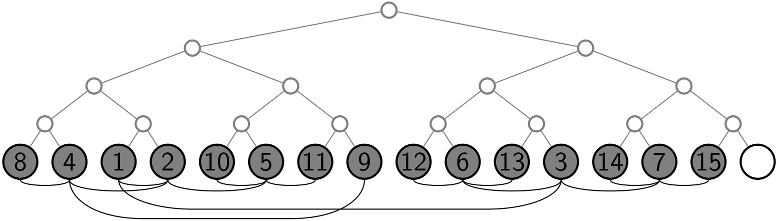

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- $OV(G, 2, \phi) = 2\sum_{i=1}^{h} a_i(\phi) \cdot i$
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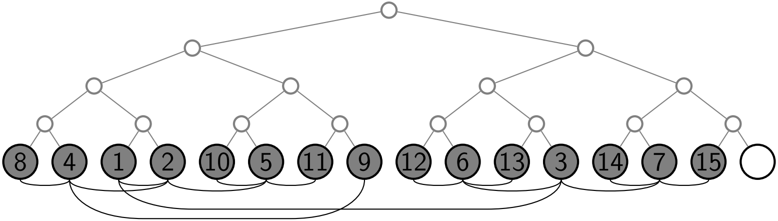

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- $a_i(\phi) = \begin{cases} s_i(\phi) - s_{i+1}(\phi) & \text{for } 1 \le i \le h-1 \\ s_i(\phi) & \text{for } i = h \end{cases}$

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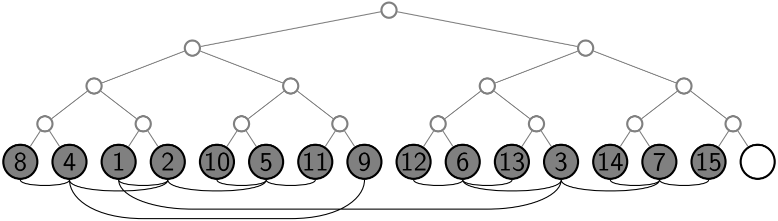

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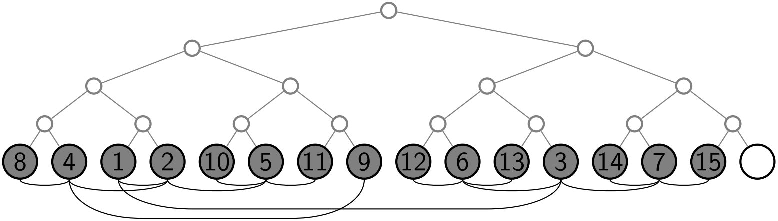
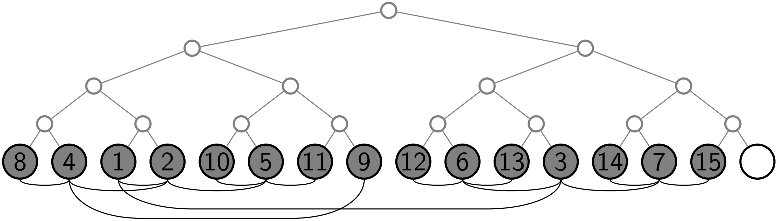


$$OV(G, 2, \phi^*) = 56$$

i	4	3	2	1
▶ a_i	1	3	5	5
s_i	1	4	9	14

Lower bound – problem transformation



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  - $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $\ldots$, $V_k \neq \emptyset$, where
  - $\cup_{i=1}^{k} V_k = V$, $V_i \cap V_j = \emptyset$ for every $i \neq j$ and
  - $|V_i| \leq \left\lceil \frac{n}{k} \right\rceil$ for all $1 \leq i \leq k$,

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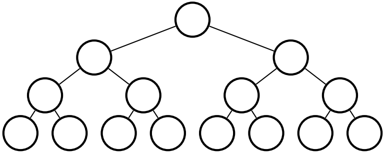
  - $|V_i| \leq \left\lceil \frac{n}{k} \right\rceil$ for all $1 \leq i \leq k$,

  such that the number of edges connecting these vertex sets

$$c(G, \mathscr{V}) := \Big| \big\{ (u, v) \in E | u \in V_i,\ v \in V_j,\ i \neq j \big\} \Big|, \tag{3}$$

  where $\mathscr{V} = \{ V_i | 1 \leq i \leq k \}$, is minimised.

Lower bound – problem transformation



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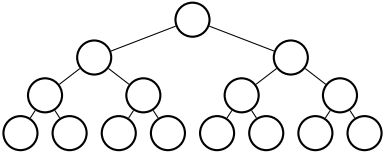

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- It is obvious that $s_i \geq c(G, \mathscr{V})$, where $k = 2^{h-i+2}$ for all $2 \leq i \leq h$ and that $s_1 = \left|E(G)\right|$.

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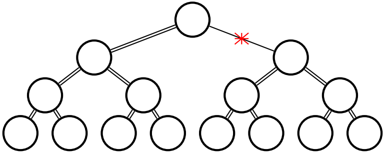

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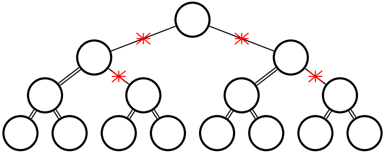
Lower bound – problem transformation



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i	4	3	2	1
a_i	1	3	5	5
s_i	1	4	9	14

, $k = 2^{3-3+2} = 4$

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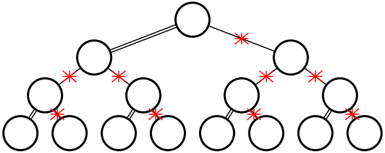
Lower bound – problem transformation



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i	4	3	2	1
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, $k = 2^{3-2+2} = 8$

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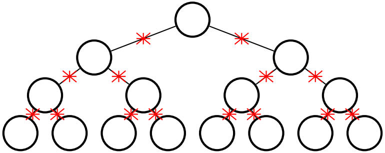
Lower bound – problem transformation



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a_i	1	3	5	5
s_i	1	4	9	14

- It is obvious that $s_i \geq c(G, \mathscr{V})$, where $k = 2^{h-i+2}$ for all $2 \leq i \leq h$ and that $s_1 = \left|E(G)\right|$.

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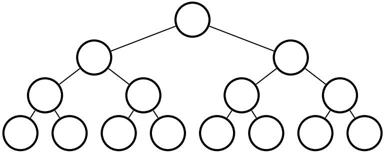

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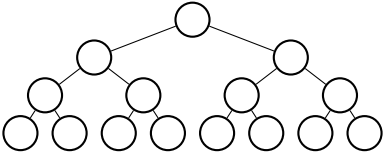

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- One component has the size $\frac{|V|+1}{k} - 1$.

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- kBPP is $\mathcal{NP}$-hard (we get the *minimum bisection problem* which is $\mathcal{NP}$-hard for $k = 2$ [Garey, Johnson 2002[6]]).

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Lower bound – problem transformation

- kBPP remains $\mathcal{APX}$-hard even if the graph is an unweighted tree with constant maximum degree [Feldmann, Foschini 2013[9]].

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Theorem (Schauer and S.)

Let $G = (V, E)$ be a binary regular tree of height $h \geq 1$ and let $k = 2^{k'}$, where $1 \leq k' \leq h$, and $\mathscr{V}^*$ an optimal k-balanced partition. Then

$$c(G, \mathscr{V}^*) = \left(3 \cdot 2^{h+1} - 2^{k'+1}\right)\left(\frac{1}{2^s - 1} - \frac{1}{(1 - 2^{-s})\,2^{sl}}\right) + 3 \cdot 2^{h-sl+1} - 2, \quad (4)$$

where $s = h - k' + 2$ and $l = \left\lfloor \frac{h+1}{s} \right\rfloor$.

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⇒ optimality in this case ✓

- $OV(G,2,\phi^*) = 56$

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Lower bound – problem transformation

i	6	5	4	3	2	1
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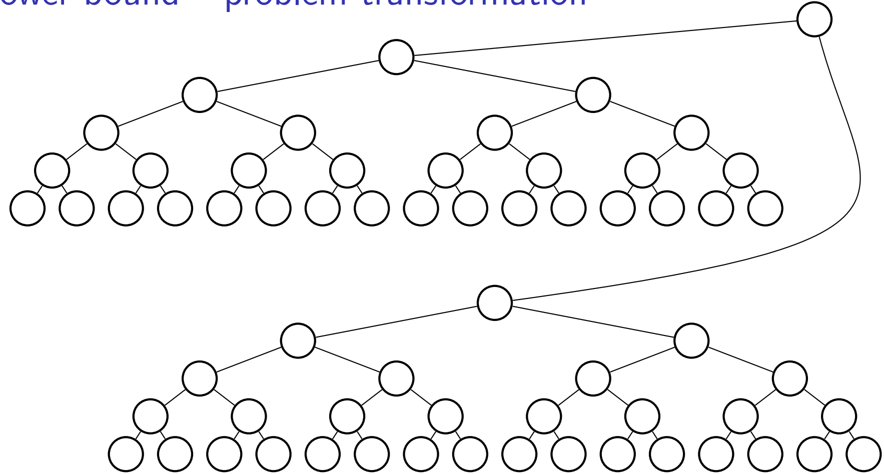
- The empirical gap between the lower and the upper bound does not exceed 1.1.

Lower bound – problem transformation



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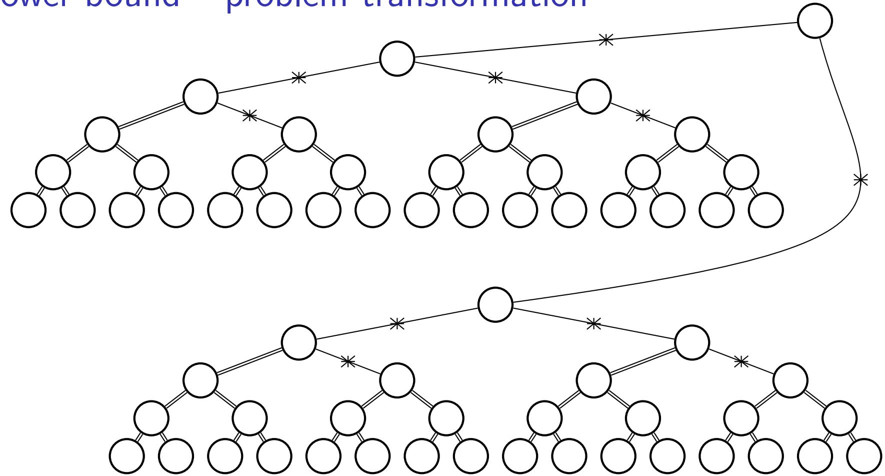

$$c(G, k, \mathscr{V}) = 10$$

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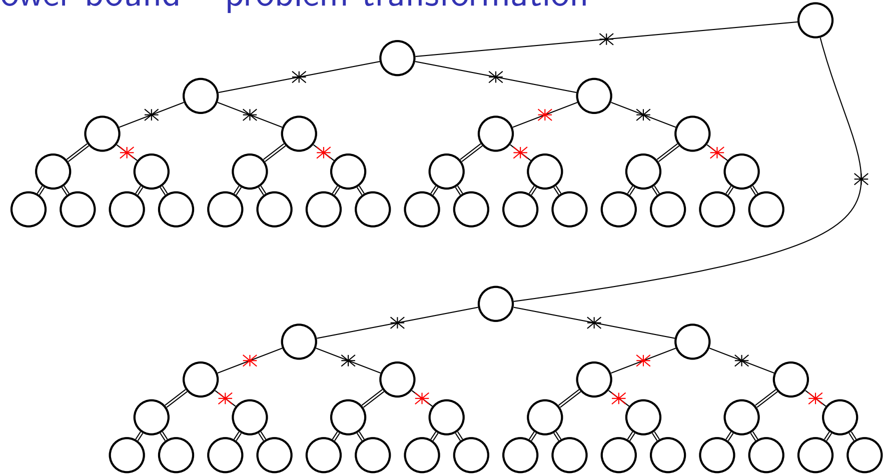
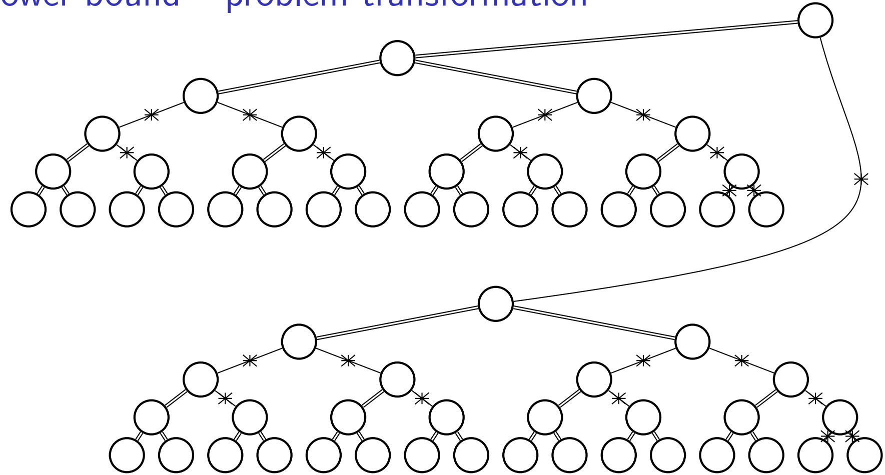


$$c(G, k, \mathscr{V}) = 10 + \color{red}{12} \color{black}{= 22}$$

Lower bound – problem transformation



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- It would be necessary to improve both the algorithm and the lower bound in order to reach the optimum.

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$$c^H(G, \mathcal{V}) = \sum_{j=1}^{k'} c(G, \mathscr{V}^{(j)}). \tag{5}$$

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- The question about the computational complexity in our special case is open.

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Thank you for your attention!